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ON THE PULSATION OF FINITE, VENTILATED CAVITIES

by

C. C. Hsu and C. F. Chen

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ABSTRACT

The phenomenon of cavity resonance is explained as an interaction between the flow of the surrounding liquid and the compression and expansion of the contained gas. In this paper, the general solution to the linearized problem of a symmetrical forebody performing symmetrical, harmonic motion, in an infinite medium, with a cavity pressure which is varying harmonically, is presented. The phenomenon of cavity resonance behind a stationary slender wedge in an infinite medium is studied in detail; the results show that self-excited pulsations at discreet frequencies are possible. It is shown that the resonance frequencies are independent of the actual size of the slender wedge; and the occurrance of resonance depends on the property of the contained gas through the equivalent Mach number M^*, which is defined as $(\rho U^2/\rho_g a_g^2)^{\frac{1}{2}}$, where ρ, ρ_g, U, a_g are the density of the liquid, density of the gas, uniform speed of the flowing liquid, and the speed of sound of the gas respectively. The first four resonance frequencies calculated by the present theory are somewhat higher than those observed experimentally by Silberman and Song (Reference 1) for a normal plate in a free jet tunnel, although the trend is quite similar. The values of M_n^*/M^*, where the subscript n denotes the nth stage resonance, compare fairly well. The theoretical and experimental values of M^* cannot be compared directly because of the difference in the forebody shape, they are, nevertheless, of comparable order of magnitude.

SYMBOLS

A	amplitude of harmonic disturbance, see Equation [4]
a_∞	speed of sound of the liquid in the uniform stream
a_g	speed of sound of the gas contained in the cavity
$\tilde{C}$	$= C_1 + i\, C_2$, a complex constant
$F_1(k/2)$, $F_2(k/2)$	defined by Equations [49] and [50]
$G(k/2)$	defined by Equation [51]
$H_o^{(2)}$	Hankel function of the second kind and zeroth order
I_n	Bessel function of nth order, $n = 0, 1, 2 \ldots$
k	reduced frequency of pulsation $= \omega \ell / U$
ℓ	cavity length
M_∞	Mach number of the uniform stream $= U/a_\infty$
M^*	equivalent Mach number $= (\rho U^2 / \rho_g a_\infty{}^2)^{\frac{1}{2}}$
m	strength of source distribution
p	local static pressure
p_c	cavity pressure
p_∞	ambient pressure
$\vec{q}$	velocity of the fluid at any point in the flow field
r	radial distance from the source
t	time

U	uniform free stream velocity parallel to the x-axis
u	x-component of the disturbance velocity
$\overline{V}_c$	cavity volume
v	y-component of the disturbance velocity
x,y	space coordinates
x_1, ξ, ξ'	dummy variable
y_b	semi-thickness of the body
y_c	semi-thickness of the cavity
λ	wave length = a_∞/ω
μ	mass of gas contained in the cavity
ρ	density of the free stream
ρ_g	density of gas contained in the cavity
σ	cavitation number = $(p_\infty - p_c)/\frac{1}{2}\rho U^2$
Φ	velocity potential
ω	frequency of pulsation

The subscript "o" and "ι", in general, denote the steady and unsteady parts of the designated quantities respectively.

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ON THE PULSATION OF FINITE, VENTILATED CAVITIES

I. INTRODUCTION

Recently, Silberman and Song (Reference 1) observed self-excited pulsations of ventilated cavities behind normal plates, symmetrical wedges, and hydrofoils in a two-dimensional free jet. Later, Song (Reference 2) analyzed the problem by neglecting the effect of the flowing stream and at the same time approximated the actual geometry by a cylindrical cavity enclosed by an annulus of quiescent water. The resonance frequency thus obtained is proportional to $[\ell n\ (R_o/r_o)]^{\frac{1}{2}}$ where R_o and r_o are the radii of the outer and inner cylinder bounding the annulus respectively. The relationship between the actual geometry involved and the annulus considered, though both are doubly connected regions, is not evident; certain emperical constants have to be determined from the experimental data. Furthermore, the result indicates that as the outer radius recedes to infinity, cavities cease to pulsate. In this paper, we shall show that when the interaction of the flowing stream and the expansion and the compression of the gas contained in the cavity is taken into account, self-excited pulsations of finite, ventilated cavities at discreet frequencies is possible in an infinite medium.

Starting from the linearized potential equation for harmonic disturbances in an unsteady, compressible flow* together with its fundamental solution in terms of Hankel functions, it

*We are ultimately interested in the noise generated by such pulsations; thus, the assumption of compressible fluid is more appropriate here.

is first shown that when the Mach number is small and the body-cavity dimension small compared to the wave length of the disturbance the governing equation becomes the Laplace's equation and the fundamental solution becomes a source of time-dependent strength. For the problem of symmetrical motions of the forebody with time-varying cavity pressures, a distribution of sources with time-varying strength has the proper symmetry required. By applying the proper boundary conditions, an integral equation results. To obtain the solution, further linearization is applied; that is, the time dependent part of the forebody motion and of the cavity pressure are assumed to be small compared with their respective mean values. The solution to the steady part is given by Tulin (Reference 3). The solution for the unsteady part is a proper combination of a particular solution and a homogeneous solution which is in the form of a travelling wave, such that the juncture condition is satisfied.

For the problem of self-excited pulsations of cavities, we have treated a cavitating flow past a stationary wedge in detail. By assuming a sinusoidal cavity pressure variation the source distribution can be obtained from the method stated above. The time-dependent volume of the cavity is obtained by a double integration of the source strength. By analyzing the gas contained in the cavity, the volume variations due to the assumed cavity pressure variation can be calculated by gas laws. These two cavity volumes, one calculated by considering the interaction of the streaming flow and the cavity, the other calculated by the gas laws, must be compatible with each other in order to have self-sustained oscillation of the cavity. From the compatibility conditions, the reduced frequencies k ($=\omega \ell_o/U$, where ℓ_o is the length of the steady cavity and U is uniform stream

speed) at which resonance may occur, and the equivalent Mach numbers M^* ($= (\rho U^2/\rho_g a_g{}^2)^{\frac{1}{2}}$, where ρ is the density of the uniform stream, and ρ_g and a_g are the density and the speed of sound of the gas inside the cavity) corresponding to these frequencies can be determined. It is found that the resonance frequencies are independent of the wedge size. The first four resonance frequencies calculated by the present theory are higher than those observed experimentally by Silberman and Song (Reference 1) for a normal plate in a free jet tunnel, although the trends are quite similar. The value of M_n^*/M^*, where the subscript n denotes the nth stage resonance, compare fairly well. The theoretical and experimental values of M^* cannot be compared directly because of the difference in the forebody shape; they are, nevertheless, of comparable order of magnitude.

II. METHOD OF ANALYSIS - LINEARIZED THEORY

A first approximation for the two-dimensional unsteady cavity flow around an obstacle in a uniform infinite stream can easily be obtained by assuming that the changes in magnitude and direction of the velocity U_∞ of the undisturbed flow, due to the presence of the body, are small; more exactly, by assuming that the squares and the higher power of the perturbation velocity can be neglected when compared with the square of the uniform stream velocity. The present analysis is essentially an extension of M. P. Tulin's work on steady two-dimensional cavity flows (Reference 3).

1. Equations of Motion

Consider, in general, the uniform two-dimensional flow of an inviscid, compressible fluid of Velocity U past a symmetric body of unit chord with blunt base so that a cavity is sustained downstream of the body. Let the x-axis be in the direction of the uniform stream and its origin be at the center of the base of the forebody, see Figure 1. Let $y_b(x,t)$ and $y_c(x,t)$ denote the body surface and cavity boundary respectively. The pressure in the cavity $p_c(x,t)$ is assumed to be a given function of time. Both the length of the cavity, $\ell(t)$ and the volumeper unit width of the cavity, are permitted to vary with time, the equation of motion is

$$\frac{\partial \vec{q}}{\partial t} + (\vec{q}\cdot\nabla)\,\vec{q} = -\frac{1}{\rho}\nabla p \qquad [1]$$

in which $\vec{q} = \vec{U} + \nabla\Phi$, Φ is the perturbation potential, p and ρ denote the pressure and density respectively in the flow field.

For irrotational motion of barotropic fluid, [1] can be written as

$$\nabla\left(\frac{\partial\phi}{\partial t} + \frac{q^2}{2} + \rho\,\frac{dp}{\rho}\right) = 0 \qquad [2]$$

The perturbation potential ϕ satisfies the following equation

$$\nabla^2\,\phi - M_\infty{}^2\,\frac{\partial^2\phi}{\partial t^2} - 2\,\frac{M_\infty}{a_\infty}\,\frac{\partial^2\phi}{\partial x\partial t} - \frac{1}{a_\infty{}^2}\,\frac{\partial^2\phi}{\partial t^2} = 0 \qquad [3a]$$

Where M $\left(= \frac{U}{a_\infty}\right)$ is the Mach number, a_∞ is the speed of sound in the undisturbed fluid. If harmonic disturbances are assumed, the fundamental source pulse solution to [3] is given by Bisplinghoff, Ashley, and Halfman (Reference 4).

$$\phi(x,y,t) = \frac{iA(\xi,\eta)U^2}{4\sqrt{1-M_\infty{}^2}}\,\exp\,i\left[\omega t + \frac{M_\infty{}^2}{1-M_\infty{}^2}\,\frac{\omega(x-\xi)}{a}\right]H_0^{(2)} \times$$

$$\left[\frac{\omega}{a(1-M_\infty{}^2)}\sqrt{(x-\xi)^2 + (1-M_\infty{}^2)(y-\eta)^2}\right]$$

in which (ξ,η) is the source point, and A is the amplitude of the harmonic disturbance. For $M_\infty \ll 1$, which is of interest here, the above expression becomes

$$\phi\,(x,y,t) = \frac{iA(\xi,\eta)}{4}\,e^{i\omega t}\,H_0^{(2)}\left(\frac{\omega r}{a_\infty}\right)$$

in which $r = \sqrt{(x-\xi)^2 + (y-\eta)^2}$. It is known that (see for example Jahnke and Emde, Reference 5)

$$H_0^{(2)}\left(\frac{r}{\lambda}\right) \to \begin{cases} \frac{2}{i\pi}\,\ell n\left(\frac{r}{\lambda}\right) & \text{for } \frac{r}{\lambda} << 1 \quad [4a] \\ \sqrt{\frac{2\lambda}{\pi r}}\; e^{i(r/\lambda - \pi/4)} & \text{for } \frac{r}{\lambda} \to \infty \quad [4b] \end{cases}$$

If the body-cavity dimension is much smaller than the wave length $\lambda = \frac{a_\infty}{\omega}$, in the near field of the body, the fundamental solution assumes the form of a source with time varying strength:

$$\Phi = \frac{A(\xi,\eta)}{2\pi}\; e^{i\omega t}\; \ell n\;(r/\lambda) \qquad [5]$$

which satisfies the governing equation for an incompressible fluid:

$$\nabla^2\;\Phi = 0 \qquad [6]$$

(Equation [6] is obtainable from [3a] by the assumption of $M^2 = 0\;(r^2/\lambda^2) = 0\;(M_\infty r/\lambda)$ which are negligible compared to 1.) As r increases, ultimately the form of Φ in [5] must be replaced by the expression [4b] which vanishes as $r^{-\frac{1}{2}}$. This behavior in the far field is due to the effect of compressibility which becomes dominant on a scale larger than the wave length of the pressure disturbance. Therefore, the boundary condition that the pressure obtained by an incompressible analysis must be zero at infinity as used by some investigators is not only unnecessary but in fact erroneous.

The pressure integral in [2] may also be replaced by p/ρ and the pressure at any instant may be obtained by integrating [2]

$$\frac{\rho_\infty - p}{\frac{1}{2}\rho U^2} = \frac{2}{U}\left(\frac{\partial}{\partial x} + \frac{1}{U}\frac{\partial}{\partial t}\right)\phi \qquad [7]$$

If the same restrictions are placed on the gas inside the cavity, i.e. $\omega \ell / a_g << 1$ $M_g (= U/a_g) << 1$, where a_g is the speed of sound of the gas. We have, essentially, an incompressible unsteady flow problem in the near field of the body-cavity region.

2. Boundary and Closure Conditions

The condition at the boundary of a typical body states simply that, over its surface, the normal component of the fluid velocity $\frac{\partial \phi}{\partial n}$ is fixed by the body's motion. If the equation of the surface of a body moving in a time-dependent fashion is

$$y_b - y_b\,(x,t) = 0 \qquad -1 < x < 0 \qquad [8]$$

then the linearized boundary condition to be satisfied along the x-axis is

$$\frac{v}{U} = \left(\frac{\partial}{\partial x} + \frac{1}{U}\frac{\partial}{\partial t}\right) y_b\,(x,t) \quad , \quad -1 \leq x \leq 0 \;,\; y = 0 \qquad [9]$$

Where v is the velocity component in y-direction. On the cavity, the pressure, $p_c(t)$, is specified, or equivalently, the cavitation number σ is specified. From [7], it follows that

$$\frac{p_\infty - p_c(t)}{\frac{1}{2}\rho\; U^2} = \sigma(t) = \frac{2}{U}\left(\frac{\partial}{\partial x} + \frac{1}{U}\frac{\partial}{\partial t}\right)\phi \qquad 0 < x \leq \ell, \quad y = 0 \qquad [10]$$

The condition that the cavity be closed at every instant can be derived from the usual kinematical requirement that at a bounding free surface the motion of the surface must coincide at any time with the motion of those fluid particles that happen to be at the surface at this time; that is,

$$\int_{-1}^{\ell} \frac{\partial}{\partial y} \phi \left(\xi, t - \frac{\ell - \xi}{U}\right) d\xi = 0 \qquad [11]$$

We further specify that the motion of the forebody be symmetric, with respect to the x-axis.

3. General Solutions

The linearized mathematical problem may be stated as follows: To find a harmonic function $\phi(x,y;t)$, symmetric with respect to the x-axis, whose gradient in the limit vanishes everywhere on a circle of sufficiently large radius about the origin which satisfies the mixed boundary conditions [9] and [10] and the closure condition [11]. A distribution of unsteady sources of strength $m(\xi, t)$ along the x-axis for $-1 \leq x \leq \ell$ produces a harmonic function with the proper symmetry. Therefore, we have

$$\phi\ (x,y,t) = \frac{1}{2\pi} \int_{-1}^{\ell} m\ (\xi, t)\ \ell n \sqrt{(x-\xi)^2 + y^2}\ d\xi \qquad [12]$$

with velocity components

$$u\ (x,y;t) = \frac{\partial \phi}{\partial x} = \frac{1}{2\pi} \int_{-1}^{\ell} \frac{m(\xi, t)}{(x-\xi)^2 + y^2} (x-\xi)\ d\xi \qquad [13]$$

$$v\ (x,y;t) = \frac{\partial\phi}{\partial y} = \frac{1}{2\pi}\int_{-1}^{\ell} \frac{m(\xi,t)}{(x-\xi)^2+y^2}\, y\, d\xi \qquad [14]$$

Here and subsequently, $\ell = \ell(t)$, It can be shown that at the x-axis

$$u\ (x,0,t) = \frac{1}{2\pi}\int_{-1}^{\ell} \frac{m(\xi,t)}{x-\xi}\, d\xi \qquad [15]$$

$$V\ (x,0,t) = \frac{1}{2}\, m(x,t) = U\left(\frac{\partial}{\partial x} + \frac{1}{U}\frac{\partial}{\partial t}\right)\begin{cases} y_b(x,t) & -1\leq x\leq 0 \\ y_c(x,t) & 0<x\leq \ell \end{cases} \qquad [16]$$

The mixed boundary condition on the x-axis will be satisfied if

$$m(x,t) = 2U\left(\frac{\partial}{\partial x} + \frac{1}{U}\frac{\partial}{\partial t}\right) y_b(x,t) \qquad -1 \leq x \leq 0 \qquad [17]$$

and

$$\left(\frac{\partial}{\partial x} + \frac{1}{U}\frac{\partial}{\partial t}\right)\int_{-1}^{\ell} m(\xi,t)\,\ell n(x-\xi)\, d\xi = \pi U\sigma(t) \quad 0 < x < \ell \qquad [18]$$

Decomposing the source distribution into steady and unsteady parts, and using the subscripts "o" and "1" to denote the steady and the unsteady part respectively

$$m(x,t) = m_b(x,t) = m_{b,o}(x) + m_{b,1}(x,t),\ \frac{m_{b,1}}{m_{b,o}} \ll 1 \quad -1 \leq x \leq 0 \qquad [19]$$

$$m(x,t) = m_c(x,t) = m_{c,o}(x) + m_{c,1}(x,t)\ \frac{m_{c,1}}{m_{c,o}} \ll 1 \qquad 0 < x \leq \ell \qquad [20]$$

and

$$\sigma(t) = \sigma_o + \sigma_1(t) \quad \left|\frac{\sigma_1}{\sigma_o}\right| \ll 1 \qquad 0 < x < \ell \qquad [21]$$

[18] becomes

$$\pi U \sigma_o = \int_{-1}^{0} \frac{m_{b,o}(\xi)}{x-\xi}\, d\xi + \int_{0}^{\ell_o} \frac{m_{c,o}(\xi)}{x-\xi}\, d\xi, \quad 0 < x < \ell_o, \qquad [22]$$

and

$$\pi U \sigma_1(t) = \left(\frac{\partial}{\partial x} + \frac{1}{U}\frac{\partial}{\partial t}\right)\left\{\int_{-1}^{0} m_{b,1}(\xi,t)\,\ell n\left|x-\xi\right| d\xi + \int_{0}^{\ell} m_{c,1}(\xi,t)\,\ell n\left|x-\xi\right| d\xi\right\},$$

$$0 < x < \ell \qquad [2\ ]$$

in which ℓ_o is the length of the steady cavity and from [17],

$$\begin{cases} m_{b,o} = 2U \dfrac{\partial y_{b,o}}{\partial x}, & [24] \\ m_{b,1} = 2U\left(\dfrac{\partial}{\partial x} + \dfrac{1}{U}\dfrac{\partial}{\partial t}\right) y_{b,1} & [25] \end{cases}$$

and it is assumed $\left|\dfrac{y_{b,1}(x,t)}{y_{b,o}}\right| \ll 1.$

The general solution for [22] is given by Tulin (Reference 3)

$$m_{c,o}(x) =$$

$$-\frac{1}{\pi^2}\sqrt{\frac{x}{\ell_o - x}}\left\{\int_0^{\ell_o}\sqrt{\frac{\ell_o-\xi}{\xi}}\;\frac{1}{x-\xi}\left[\pi U\sigma_o - 2U\int_{-1}^{0}\frac{dy_{b,o}(x_1)}{dx_1}\,\frac{dx_1}{\xi - x_1}\right]d\xi\right\} \qquad [26]$$

For the solution of [23], we first rewrite the equation as

$$\left(\frac{\partial}{\partial x} + \frac{1}{U}\frac{\partial}{\partial t}\right)\int_0^{\ell} m_{c,1}(\xi,t)\ \ell n\ \left|x-\xi\right|\ d\xi = \pi U\sigma_1(t)$$

$$-\left(\frac{\partial}{\partial x} + \frac{1}{U}\frac{\partial}{\partial t}\right)\int_{-1}^{0} m_{b,1}\ (\xi,t)\ \ell n\ \left|x-\xi\right|\ d\xi \qquad [23a]$$

It is noticed that any arbitrary travelling wave solution can be added to the particular solution without disturbing the function given on the right hand side of the equation, i.e. this equation admits homogeneous solutions of the following form

$$m^h_{c,1}(x,t) = -\frac{1}{\pi^2\sqrt{x(\ell-x)}}\int_0^{\ell}\frac{\sqrt{\xi(\ell-\xi)}}{(x-\xi)}\ f'\ (x-Ut)\ d\xi \qquad [26]$$

which is the solution of the following integral equation

$$\int_0^{\ell} m^h_{c,1}\ (\xi,t)\ \ell n\ \left|x-\xi\right|\ d\xi \quad f\ (x-Ut). \qquad [27]$$

A particular solution of [23a] is

$$m^p_{c,1}\ (x,t) = \frac{g(t)}{\sqrt{x(\ell-x)}} \qquad [28]$$

where

$$g(t) = \frac{U\int\left[\pi U\sigma_1(t)-\left(\frac{\partial}{\partial x}+\frac{1}{U}\frac{\partial}{\partial t}\right)\int_{-1}^{0} m_{b,1}(\xi,t)\ell n\left|x-\xi\right| d\xi\right] dt}{\int_0^{\ell}\frac{\ell n\ x-\xi\ d\xi}{\sqrt{\xi(\ell-\xi)}}} \qquad [29]$$

which may be verified by substitution of [28] and [29] into [23a] and noting that

$$\int_0^{\ell} \frac{1}{\sqrt{\xi(\ell-\xi)}} \frac{1}{x-\xi} \, d\xi = 0.$$

By contour integration, it can be shown that

$$\int_0^{\ell} \frac{\ell n |x-\xi| \, d\xi}{\sqrt{\xi(\ell-\xi)}} = \pi \, \ell n \frac{\ell}{4} \qquad [30]$$

we note then the particular solution becomes singular at $\ell = 4$. The solution given here then is valid only for $\ell \neq 4$. This point will be discussed further in a later section.

The unsteady source strength is obtained by a combination of the homogeneous and the particular solutions, then requiring that the juncture condition holds. This, however, does not lead to a unique solution in general since there is an arbitrary function involved. For simple harmonic oscillations of the body and/or of the cavity pressure, we may assume

$$F'(x-Ut) = \text{Re } \tilde{C} \, e^{i\omega(t-x/U)} \qquad [31]$$

where $\tilde{C}$ is a complex constant

$$\tilde{C} = C_1 + i \, C_2 \, , \qquad [32]$$

and Re denotes the real part. Then the juncture condition is sufficient to determine the constants C_1 and C_2. The general solution of [23] is

$$m_{c,1} = - \frac{1}{\pi^2 \sqrt{x(\ell - x)}} \left\{ - \pi^2 g(t) + \int_0^{\ell} \frac{\sqrt{\xi(\ell-\xi)}}{(x-\xi)} \operatorname{Re} \tilde{C} e^{i\omega(t-\xi/U)} d\xi \right\} \quad [33]$$

The cavity shape may now be found:

$$y_c(x,t) = y_b(0,t) + \frac{1}{2U}\left[\int_0^x m_{c,o}(\xi)d\xi + \int_0^x m_{c,1}(\xi, t - \frac{x-\xi}{U})\, d\xi\right] \quad [34]$$

The closure condition [11] becomes, for $\ell < \ell_o$

$$y_c(\ell,t) = y_b(0,t) + \frac{1}{2U}\left[\int_0^{\ell} m_{c,o}(\xi)d\xi + \int_0^{\ell} m_{c,1}(\xi, t - \frac{\ell-\xi}{U})\, d\xi\right] = 0 \quad [35]$$

for $\ell > \ell_o$

$$y_{b,1}(0,t) = - \frac{1}{2U} \int_0^{\ell} m_{c,1}(\xi, t - \frac{\ell-\xi}{U})\, d\xi \quad [36]$$

since

$$y_{b,o}(0) = - \frac{1}{2U} \int_0^{\ell_o} m_{c,o}(\xi)\, d\xi \quad [37]$$

The volume of the cavity per unit span is

$$\overline{V}_c(t) = \frac{1}{U} \int_0^{\ell} dx \int_0^{\ell} m_{c,o}(\xi) + m_{c,1}(\xi, t - \frac{x-\xi}{U})\, d\xi \,. \quad [38]$$

III. CAVITY RESONANCE

To analyze the cavity resonance problem, we focus our attention to the problem of a stationary wedge with a trailing cavity maintained by ventilation. It is assumed that the cavity pressure is oscillating about a mean pressure; the resonance condition corresponds to this case when such oscillations are possible with the mass of gas within the cavity being constant.

Equation [38], together with Equations [26] and [33], gives the response of the cavity volume to the time-varying cavity pressure. At the resonance condition, this volume variation must be compatible with those produced by the gas inside the cavity.

Let μ be the constant mass and $\rho_g(t)$ be the time-dependent density of the gas in the cavity. The volume of the cavity is

$$\overline{V}_c(t) = \frac{\mu}{\rho_g(t)} \quad .$$

The variation of the volume, $\overline{V}_{c,1}(t)$, due to a small variation of pressure is

$$\overline{V}_{c,1}(t) = - \left(\frac{\mu}{\rho_g}\right) \frac{1}{\rho_g} \frac{d\rho_g}{dp_c} dp_c \qquad [39]$$

Assuming that the pressure variations are not too rapid, the change of the density with respect to the pressure may be considered isentropic, then $\frac{d\rho_g}{dp_c} = \frac{1}{a_g^{\,2}}$. Equation [39] becomes, when the variation in cavity pressure is written in terms of the cavitation number

$$\frac{\overline{V}_{c,1}(t)}{\overline{V}_{c,o}} = \frac{1}{2} \frac{\rho U^2}{\rho_g a_g^{\,2}} \sigma_1(t) = \frac{1}{2} M^{*2} \sigma_1(t) \qquad [40]$$

in which the equivalent Mach number M^* is defined as

$$M^{*2} \equiv \frac{\rho U^2}{\rho_g a_g^{\,2}} . \qquad [41]$$

Equating the volume $\overline{V}_{c,1}(t)$ calculated by [40] to that obtained by [38], the condition at which resonance occurs can be determined.

Let the wedge profile be described by

$$y_b(x) = y_b(0)\,(1+x), \qquad [42]$$

and the pressure in the cavity by

$$\sigma(t) = \sigma_o\,(1 + \epsilon \sin \omega t) \qquad [43]$$

where $\epsilon\sigma_o$ and ω are the amplitude and frequency of the fluctuations of the cavity pressure respectively. Then

$$\varepsilon(t) = -\frac{\pi U^2}{\omega} \epsilon\, \sigma_o \cos \omega t . \qquad [44]$$

To apply the juncture condition the integral in [35] at $x = 0$ can be evaluated by using the angular transformation $\xi = \sin^2 \frac{\theta}{2}$ and the Jacob's expansion

$$e^{i\,k/2 \cos\theta} = J_o(k/2) + 2 \sum_{n=1}^{\infty} i^n J_n(k/2) \cos n\theta .$$

We obtain for the unsteady source strength

$$m_{c,1} = -\frac{\epsilon\,\sigma_o U\ell}{k}\ \frac{1}{\ell n\ \frac{\ell}{4}}\ \frac{1}{\sqrt{x(\ell-x)}}\left\{\cos\omega t + \frac{2}{\ell\pi(J_o^{\,2}+J_1^{\,2})}\int_0^{\ell}\frac{\sqrt{x'(\ell-x')}}{x-x'}\ \mathrm{Re}\ \tilde{C}\ e^{i\omega(t-x'/U)}\ dx'\right\} \qquad [45]$$

where $\tilde{C} = C_1 + i\ C_2$

$$= (J_o \sin\frac{k}{2} + J_1 \sin\frac{k}{2}) + i\ (J_o \sin\frac{k}{2} - J_1 \cos\frac{k}{2}) \qquad [46]$$

$k = \frac{\omega\ell}{U}$, the reduced frequency,

and the arguments of the Bessel Functions J_o and J_1 are k/2.

To apply the closure condition to find the relationship between the unsteady cavity length $\ell(t)=\ell_o+\ell_1(t)$ and the unsteady cavity pressure, we first use [37] in [35] to obtain

$$-\int_{\ell_o-|\ell_1(t)|}^{\ell_o} m_{c,o}(\xi)\ d\xi + \int_0^{\ell} m_{c,1}(\xi, t - \frac{\ell-\xi}{U})\ d\xi = 0\ . \qquad [47]$$

Consistent with the assumptions made already, we assume $\frac{|\ell_1(t)|}{\ell_o} \ll 1$. The first integral can be evaluated by replacing all x's in the $m_{c,o}$ expression [26] by ℓ_o except the singular part $(\ell_o-x)^{-\frac{1}{2}}$. The result is

$$\text{const.}\ \sqrt{\ell_1(t)}\ .$$

The second integral is clearly of order ϵ. Equation [47] gives the result that

$$|\ell_1(t)| \sim O(\epsilon^2) ,$$

and it may be neglected in our present theory.

In view of the above result, to obtain the additional cavity volume due to the unsteady pressure, we may replace ℓ by ℓ_o in the expression [45] for $m_{c,1}$ including that in the reduced frequency k and in the upper limit of the integral for the cavity volume:

$$\overline{V}_{c,1}(t) = \int_0^{\ell_o} dx \int_0^x m_{c,1}\left(\xi, t - \frac{x-\xi}{U}\right) d\xi .$$

This integration can be carried out once the angular transformation $\xi = \sin^2 \frac{\theta}{2}$ and the Jacobi's expansions are used. The result is

$$\overline{V}_{c,1}(t) = \frac{\epsilon\sigma_o U\pi\ell^2}{k^2} \frac{1}{\ell n \frac{\ell}{4}} \left\{F_1(k/2) \sin \omega t - F_2(k/2) \cos \omega t\right\} \quad [48]$$

in which

$$F_1(k/2) = J_o \cos\frac{k}{2} - 1 + G(k/2) \left[J_o \sin\frac{k}{2} + J_1 \cos\frac{k}{2}\right] \quad [49]$$

$$F_2(k/2) = J_o \sin\frac{k}{2} - G\left(\frac{k}{2}\right) \left[J_o \cos\frac{k}{2} - J_1 \sin\frac{k}{2}\right] \quad [50]$$

$$G(k/2) = \frac{1}{J_o^2 + J_1^2} \left[J_o J_1 + \frac{4}{k} \sum_{n=1} n J_n^2\right] \quad [51]$$

and all the arguments of the Bessel functions are k/2. The additional volume given by [40] becomes

$$\frac{\overline{V}_{c,1}}{\overline{V}_{c,o}} = \frac{1}{2} M^{*2} \epsilon \sigma_o \sin \omega t \qquad [52]$$

For small values of σ_o, Tulin's result (Reference 37) gives

$$\overline{V}_{c,o} = \frac{\pi \ell_c^{\ 2}}{8} \sigma_o \qquad [53]$$

Combining [53] with [48], and comparing with [52], we require that

$$\begin{cases} F_2 \left(\frac{k_n}{2}\right) = 0 & [54] \\ M^{*2} = \frac{1}{\sigma_o \ell n \frac{\ell o}{4}} \frac{16}{k^2} F_1 \left(\frac{k_n}{2}\right) \qquad \ell_o \neq 4 & [55] \end{cases}$$

The subscript n denotes the nth root of F_2 $(k/2)$ for which

$$F_1 \ (k_n/2) \left[\ell n \ \left(\frac{\ell_o}{4}\right)\right]^{-1} > 0 \ .$$

This means that of $\ell_o > 4$, resonance may occur when F_1 $(k/2) > 0$. Whereas for $\ell_o < 4$, the resonance condition becomes $F_1 (k/2) < 0$. For cavity lengths in the neighborhood of four chord lengths of the wedge, M^* becomes arbitrarily large. Since the value of $\frac{1}{k^2} F_1 \ (\frac{k}{2})$ is highly damped as k becomes large, only high modes of resonance may occur in the vicinity of $\ell_o = 4$. This result is shown in Figure 3, which will be discussed presently.

From [50] and [51], it is seen that

$$\left.\begin{aligned} F_2\ (k/2) &\rightarrow k^2 \\ \frac{16}{k^2} F_1\ (k/2) &\rightarrow 3 \end{aligned}\right\} \qquad \text{as } k \rightarrow 0$$

which means that for the case $\ell_o > 4$, $k = 0$ is an admissible root. But this corresponds to $\omega = 0$ and $\overline{V}_{c,1} = 0$, which reduces to the steady case. It is seen from [54] that the resonance frequencies are independent of the wedge size. The function $\frac{1}{k} F_2\ (k/2)$ together with $\frac{16}{k^2}\ F_1\ (\frac{k}{2})$ are plotted in Figure 2. It is seen that for $\ell_o > 4$ the first non-zero root as well as the third, the fifth, and so on of F_2 are inadmissible because the corresponding value of F_1 is negative. The <u>inadmissible</u> roots for the case of $\ell_o > 4$ become the <u>admissible</u> roots for the case of $\ell_o < 4$. The first four admissible roots of $F_2(k/2)$ for both $\ell_o > 4$ and $\ell_o < 4$ and the corresponding values of $\frac{16}{k^2}\ F_1\ (\frac{k}{2})$ are given in Table I.

Silberman and Song (Reference 1) have presented experimental data of pulsating cavities behind a 1/8" normal plate in a free jet tunnel. They have observed cavity pulsation which are characterized by the number of waves appearing on the cavity. They have designated the pulsation with n waves as the nth stage pulsation. It is conjectured here that the present nth k would give a similar cavity shape. Since the indefinite integral for the cavity shape is extremely difficult to evaluate, it has not

been attempted here. From these data, the reduced frequency and the equivalent Mach number corresponding to each of the stages may be calculated. In calculating the density of the contained air at the cavity pressure, we have assumed that the isothermal equation of state holds. These values are also given in Table I. The reduced resonance frequencies are given as a range because the experimentally observed cavity length varied. The values of M_n^*/M_1^* have also been calculated and presented in the Table I.

Since in the theory the cavity length is measured with respect to the chord of the wedge, the experimental results obtained from a 1/8" normal plate, in a strict sense, cannot be compared with the theoretical results. However, it is of interest to see the orders of magnitude of the values obtained. With this in mind, we make the following comparison. The theoretically predicted resonance frequencies are, in general, higher than those observed experimentally. The values of M_n^*/M_1^* seem to be in fairly good agreement. The theoretical results are for an infinite medium. The effect of the free surfaces bounding the jet in the case of the experiment would probably alter the theoretical results to some extent. From the results obtained, however, it is seen that the presence of the free jet surface is not a necessary condition for the occurrance of self-excited cavity pulsation.

In terms of numerical examples, a 15°-wedge with a trailing cavity at a mean cavitation number of 0.07, the equivalent Mach number of the first stage pulsation is 1.25 as compared to the experimental value of 2.26 for a normal plate in a free jet with a cavitation number of 0.0735. It is interesting to note that these are of the same order of magnitude in spite of

the different forebody configurations and the free surface conditions present in the experimental case. In the presence of free surfaces, it is known that the cavity length is shorter than that in an infinite medium (Reference 6). It is conjectured here that the effect of the free surface is to increase the value of the equivalent Mach number because of the decrease in cavity length. This 15°-wedge with $\sigma_{o} = 0.07$, may experience the first stage self-induced pulsation at a forward speed of 50 ft/sec.

In Figure 3, we have shown the locus of the points for which resonance of the first three stages are possible for a 15°-wedge travelling at 5 feet below the free surface. (In this example, we assume the presence of the free surface does not alter the results obtained above.) Two families of loci appear, one for $\ell_{o} < 4$, one for $\ell_{o} > 4$. Cavity resonance is only possible when the values of $\frac{p_c}{p_\infty}$ and U correspond to a point on one of these curves. It is interesting to note that the family of loci for $\ell_{o} > 4$ exhibit a minimum speed below which no resonance corresponding to that state or lower may occur. We may trace the history of the unsteady motion of a cavity behind a 15°-wedge as follows. Suppose $p_c/p_\infty = 0.75$. As speed increases from zero, first stage oscillation may be encountered at 30 ft/sec. As the speed is increased further the flow becomes stable until at 50 ft/sec, second stage oscillation appears. Then as the speed is increased still further, higher and higher modes of oscillation appear. At 60 ft/sec, which corresponds to the critical length $\ell_{o} = 4$, no oscillations may appear. Between U = 60 and 66 ft/sec, the mode of oscillation starts high and

decreases to n = 1. Beyond 65 ft/sec, oscillations of the cavity are again not likely. It is in the range of speeds from 50 to 66 ft/sec that self-excited oscillations would seem most likely to appear. The results presented in this Figure show that cavity resonance can only occur for a finite range of speeds.

IV. SUMMARY AND CONCLUSIONS

The theory presented in this report gives a simple method for determining approximately the characteristics of two-dimensional unsteady cavity flows about slender symmetrical bodies in a uniform infinite stream. The present paper is intended to throw some light on the physical mechanism of the pulsation of the cavities. The account is mainly theoretical, but a brief reference is made to some recent experiments on the instability of ventilated cavities. The compressibility of the gas inside the cavity is pointed out to be an essential factor in this aspect of cavitation. The case of ventilated cavity flow about a thin wedge has been discussed in detail.

The important results obtained in this study may be summarized as follows:

1. Pulsation of finite, ventilated cavity is possible in an infinite medium under ordinary conditions on the speed and cavitation number. Within the framework of the analysis made, the existence of a free surface is not a prerequisite for self-excited pulsation.

2. The occurrance of resonance depends on the equivalent Mach number M^* which involves not only the property of the flowing liquid but also that of the contained gas in the cavity: $M^* = (\rho/\rho_g)^{\frac{1}{2}} U/a_g$. The reduced resonance frequencies, $k = \omega \ell_o/U$, are infinitely many and they are discreet.

3. The resonance frequencies calculated for a cavity created behind a two-dimensional wedge is independent of the wedge size. The equivalent Mach number, however, depends on the length of the cavity.

4. Calculations made for stationary wedges show that there exists a critical cavity length of $\ell_o = 4$. The resonance frequencies and the equivalent Mach numbers for cases in which $\ell_o > 4$ are quite different from those for $\ell_o < 4$. Furthermore, the theory shows that cavity resonance can only occur within a finite speed range.

5. It is conjectured that the nth resonance frequency would give a cavity shape with n waves similar to those observed by Silberman and Song (Reference 1).

6. The calculated resonance frequencies and equivalent Mach numbers are qualitatively comparable to those observed experimentally by Silberman and Song (1) in a free jet tunnel.

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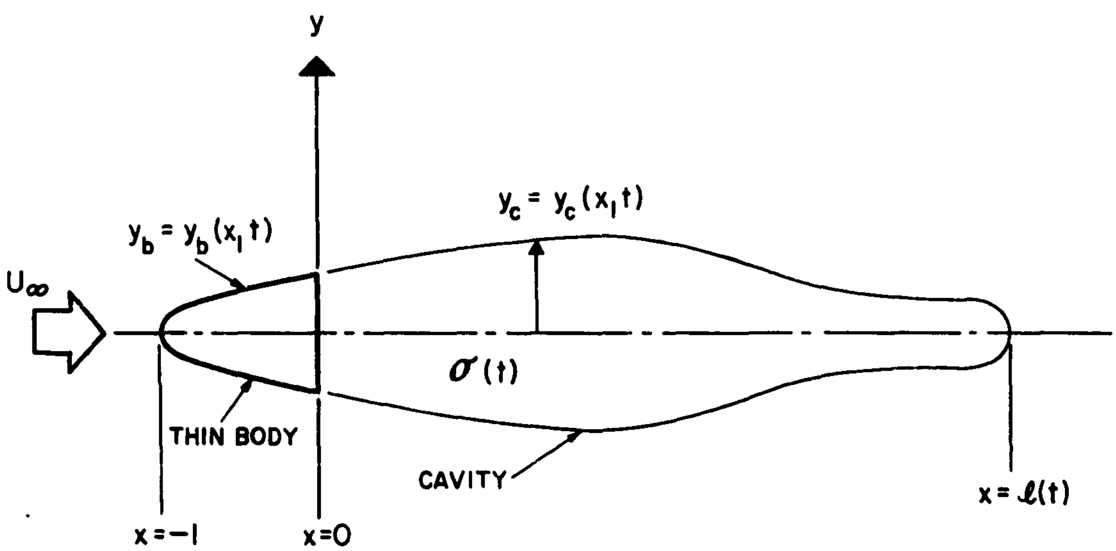

FIGURE 1 – SCHEMATIC OF UNSTEADY CAVITATION FLOW PAST A THIN BODY

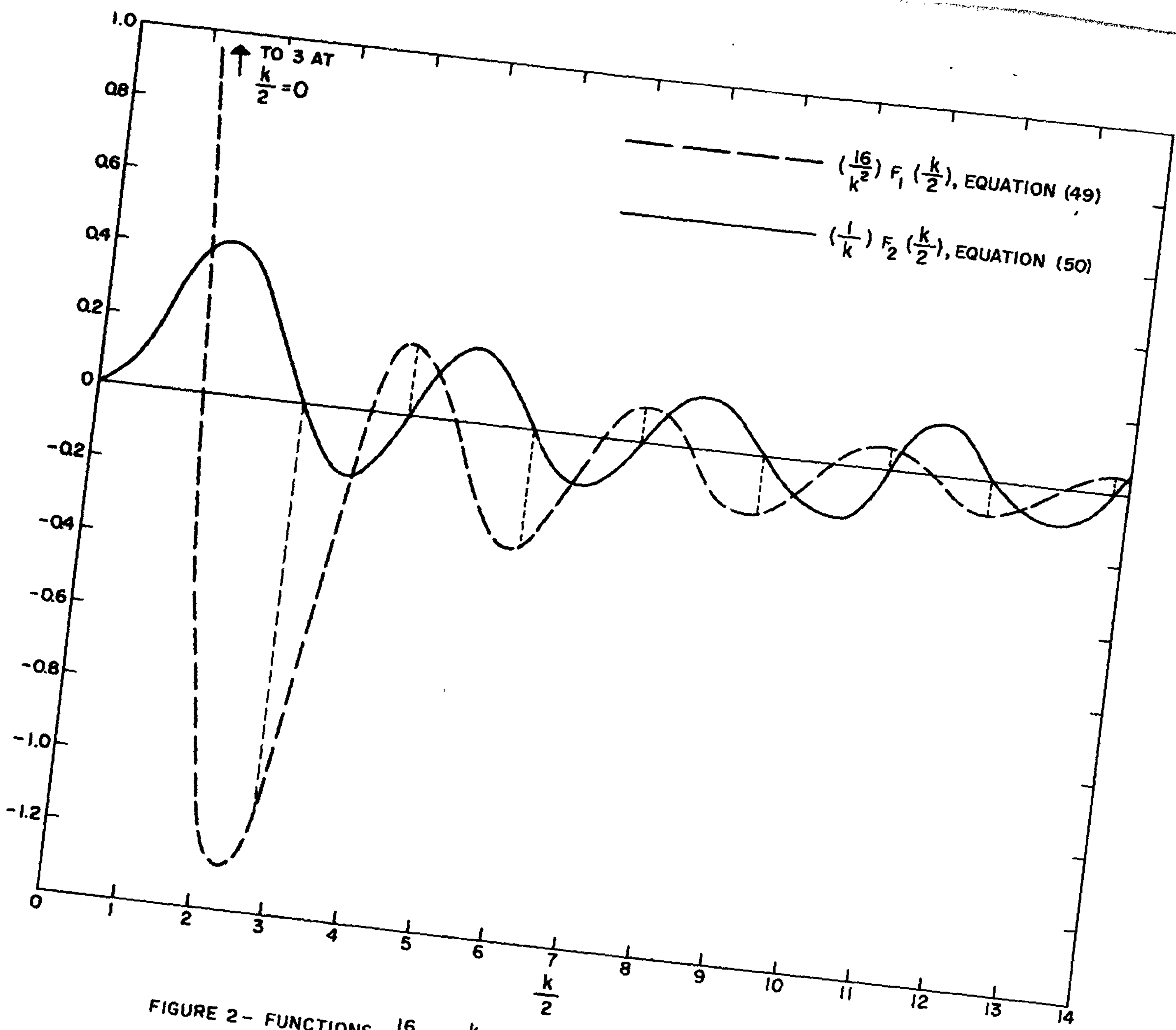

FIGURE 2 – FUNCTIONS $\frac{16}{k^2} F_1 (\frac{k}{2})$ AND $\frac{1}{k} F_2 (\frac{k}{2})$ SEE EQUATIONS (49) AND (50)

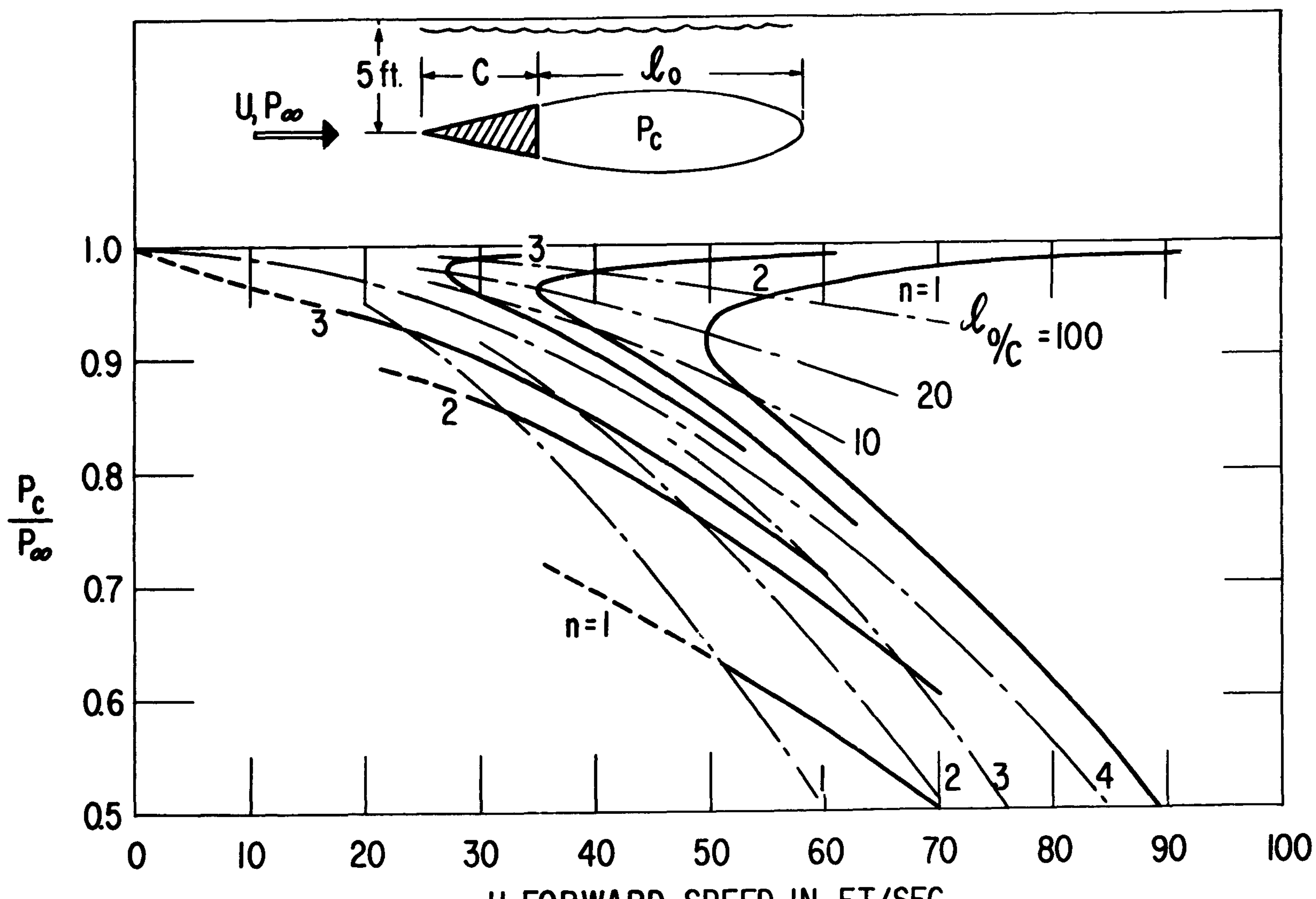

FIGURE 3– CAVITY RESONANCE CONDITIONS FOR A 15° WEDGE AT 5 FEET DEPTH

TABLE I. THEORETICAL AND EXPERIMENTAL RESULTS
(Experiments of Silberman and Song)

n		1	2	3	4
k_{theory}	$\ell_o > 4$	8.4	14.8	21.5	27.4
	$\ell_o < 4$	5.5	11.8	18	24.2
$k_{exp}(\frac{1}{8}$ " normal plate)		4.5-6.1	9.9-13.0	14.2-19	17.1-22.9
$\frac{16}{k^2} F_1(\frac{k}{2})$	$\ell_o > 4$	0.192	0.097	0.055	0.044
	$\ell_o < 4$	-1.11	-0.32	-0.16	-0.095
$\frac{M_n^*}{M_1^*}$ (theory)	$\ell_o > 4$	1	0.505	0.29	0.23
	$\ell_o < 4$	1	0.536	0.38	0.30
$M^*_{exp}(\frac{1}{8}$" normal plate)		2.26	1.25	0.819	0.631
$\frac{M_n^*}{M_1^*}$ (exp)		1	0.55	0.36	0.28

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